Algebraic Curves : Solutions sheet 1

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Exercise 1. Let R be a ring and $I \subsetneq R$ a proper ideal of R. A ring is called *reduced* if it does not contain any (non-zero) nilpotent element.

1. Show there is a bijection between the ideals of R/I and the ideals of R containing I.
2. Show that I is maximal if, and only if, R/I is a field.
3. Show that I is prime if, and only if, R/I is a domain.
4. Show that I is radical if, and only if, R/I is reduced.

Main idea.
1. Recall a similar statement for groups and the way to prove it.
2. If $\mathfrak{m}$ is a maximal ideal, then if $\mathfrak{m} \subsetneq J$ with J bigger ideal, then $J = R$ and in particular $1 \in J$.
3. See the parallel in the definitions : in a prime ideal $\mathfrak{p}$, for all $a, b \in R$ such that $ab \in \mathfrak{p}$, $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. In a domain, for all $a, b \in R$ such that $ab = 0$, $a = 0$ or $b = 0$.
4. Recall the definitions of radical ideals and reduced rings and try to parallel them.

Solution 1.
In general for this exercise, you can use the quotient map

$$\pi : R \longrightarrow R/I$$

which is a surjective ring morphism. It is helpful to transfer from one context to another.

1. The method consists of defining two maps of sets which are inverse to each other. The maps are given by the following :

$$\{I \subseteq I' \subseteq R \text{ ideal }\} \longrightarrow \{J \subseteq R/I \text{ ideal }\}$$

$$I' \longmapsto \pi(I')$$

$$\pi^{-1}(J) \hookleftarrow J$$

   We need to check :

   - Well-defined :
     - if $I \subseteq I' \subseteq R$ ideal, then $\pi(I')$ ideal in R/I

- if J ideal in R/I, $\pi^{-1}(J)$ also ideal of R containing I.

- Inverse : check that for $I \subseteq I' \subseteq R$, $\pi^{-1}(\pi(I')) = I$. The inclusion $\pi^{-1}(\pi(I')) \subseteq I$ requires crucially the hypothesis that $I \subset I'$.

2. Consider $x \notin \mathfrak{m}$. Then $\mathfrak{m} + (x) = R$, so there exists $a \in R, m \in \mathfrak{m}$ such that $1 = m + ax$. The inverse of x is given by a. This way, we obtain an inverse for each non-zero element of $R/\mathfrak{m}$ which makes it a field. The proof of the converse is similar. We can also use the previous statement, then it is very direct but less intuitive.

3. If I is prime, let $a, b \in R/I$ such that $ab = 0$. It means that there are $a', b' \in R$ such that $a'b' \in I$ (using that the quotient map is surjective and a ring morphism). Then, I is prime so either a' or b' is in I, and so either a or b is zero in R/I. The proof of the converse is similar.

4. The same reasoning works.

Exercise 2. Let R be a ring, S a multiplicative subset of R and M an R-module. Define the localization of M as the set $S^{-1}M = \{\frac{m}{s},\ m \in M,\ s \in S\}/\sim$, where: $\frac{m}{s} \sim \frac{m'}{s'} \Leftrightarrow \exists t \in S,\ t \cdot (s' \cdot m - s \cdot m') = 0$

1. Check that the relation $\sim$ defined above is indeed an equivalence relation and that $S^{-1}M$ is an $S^{-1}R$-module.

2. Show that localization preserves short exact sequences i.e. if

$$0 \to L \to M \to N \to 0$$

is an exact sequence of R-modules, then

$$0 \to S^{-1}L \to S^{-1}M \to S^{-1}N \to 0$$

is an exact sequence of $S^{-1}R$-modules.

3. Let $I \subset R$ be an ideal of R. Show that $S^{-1}I$ is an ideal of $S^{-1}R$ and that we have an isomorphism of rings

$$S^{-1}R/S^{-1}I \simeq (S/I)^{-1}(R/I),$$

where S/I denotes the image of S in R/I.

Solution 2.

1. - To check that this defines an equivalence relation, you need reflexivity, symmetry and transitivity. The transitivity follows by the following : let $\frac{m_1}{s_1} \sim \frac{m_2}{s_2}$ and $\frac{m_2}{s_2} \sim \frac{m_3}{s_3}$. There are $t_1, t_2 \in R$ such that :

$$t_1(s_2m_1 - s_1m_2) = 0$$

$$t_2(s_3m_2 - s_2m_3) = 0$$

Multiplying the last equation by t_1m_1, substituting $t_1s_2m_1$ using the first equation, we get :

$$t_2m_2t_1(s_3m_1 - s_1m_3) = 0$$

which asserts that $\frac{m_1}{s_1} \sim \frac{m_3}{s_3}$

- $S^{-1}M$ is an $S^{-1}R-$module. The action is naturally defined using the $R-$module structure of M :

$$\frac{x}{s_x} \in S^{-1}R, \frac{m}{s_m} \in S^{-1}M, \ \frac{x}{s_x} \cdot \frac{m}{s_m} := \frac{x \cdot m}{s_x s_m}$$

  We need to check that it does not depend on the choice of representative in the equivalence class of $\frac{x}{s_x}$ and $\frac{m}{s_m}$. Let's see for $\frac{m}{s_m}$. Let $\frac{m'}{s_{m'}} \sim \frac{m}{s_m}$. There exists t_m such that $t_m(s_{m'}m - s_m m') = 0$. Acting on both side by x, we get

$$t_m(s_{m'}m \cdot x - s_m m' \cdot x) = 0$$

  Then $t = t_m s_x$ gives $\frac{x \cdot m}{s_x s_m} \sim \frac{x \cdot m'}{s_x s_{m'}}$.
  The other properties of scalar multiplication (like distributive law) can be easily deduced from the definition.

2. There is a preliminary step for this proof, which is to understand that localisation is functorial, i.e. a morphism of $R-$module $f : L \to M$ induces a ring morphism $S^{-1}f : S^{-1}L \to S^{-1}M$, defined by

$$\frac{x}{s} \longmapsto \frac{f(x)}{s}$$

   Check that it is well-defined. Now, it is clear that

   - $\mathsf{ker} f = 0$ implies $\mathsf{ker}\ S^{-1}f = 0$.
   - f surjective implies $S^{-1}f$ surjective : let $\frac{m}{s} \in S^{-1}M$. f surjective implies that there exists $l \in L$, $m = f(l)$. Then $\frac{m}{s} = S^{-1}f(\frac{l}{s})$
   - The same works for exactness in the middle of the short exact sequence.

3. We can use the previous question with the short exact sequence :

$$0 \to I \to R \to R/I \to 0$$

   Hence, the following sequence is exact of $S^{-1}R-$module :

$$0 \to S^{-1}I \to S^{-1}R \to S^{-1}(R/I) \to 0$$

   We get directly that $S^{-1}I$ is a $S^{-1}R-$submodule of $S^{-1}R$, which is precisely the definition of an ideal. It also described our quotient as $S^{-1}(R/I)$. It remains to observe that if for $s, s' \in S$, if $s = s' + i$ for $i \in I$ then for any $\bar{r} \in R/I$, $\frac{\bar{r}}{s} \sim \frac{\bar{r}}{s'}$. Indeed, $\bar{r}(s' - s) = \bar{r}i = 0$ in R/I. Thus, $S^{-1}R/I \cong (S/I)^{-1}R/I$ where S/I is a multiplicative subset of R/I.

Exercise 3. Show that, for any exact sequence of finite-dimensional vector spaces:

$$0 \to V_1 \to V_2 \to \ldots \to V_n \to 0$$

the following relation holds:

$$\sum_{i=1}^{n} (-1)^i \cdot dim(V_i) = 0$$

Solution 3. We give a name to the maps in the sequence as follows

$$0 \to V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \dots \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} 0$$

By rank theorem and definition of exact sequences,

$$\begin{aligned} \mathsf{dim}\ V_i &= \mathsf{dim\ ker}\ d_i + \mathsf{dim\ im}\ d_i \\ \mathsf{dim}\ V_i &= \mathsf{dim\ ker}\ d_i + \mathsf{dim\ ker}\ d_{i+1} \end{aligned}$$

The following telescoping serie appears :

$$\begin{aligned} \sum_{i=1}^n (-1)^i\ \mathsf{dim} V_i &= \sum_{i=1}^n (-1)^i (\mathsf{dim\ ker}\ d_i + \mathsf{dim\ ker}\ d_{i+1}) \\ \sum_{i=1}^n (-1)^i\ \mathsf{dim} V_i &= -\mathsf{dim\ ker}\ d_1 + (-1)^n\ \mathsf{dim\ im}\ d_n \\ \sum_{i=1}^n (-1)^i\ \mathsf{dim} V_i &= 0 \end{aligned}$$

because both $\mathsf{ker}\ d_1$ and $\mathsf{im}\ d_n$ are contained in (0).

Exercise 4.

1. Let k be an infinite field, $F \in k[X_1, ..., X_n]$. Suppose $F(a_1, ..., a_n) = 0$ for all $a_1, ..., a_n \in k$. Show that $F = 0$.

2. Give an example showing that a similar result doesn't hold for a finite field.

Main idea.

1. Use induction.

2. Use Fermat's little theorem.

Solution 4.

1.
   - Base case : if $F \in k[X]$ is of degree $n \geq 1$, it has at most n roots so if k is infinite, there are $a \in k$, $f(a) \neq 0$. If f is constant, it is clear that the hypothesis implies $f = 0$.
   - Induction step : assume the statement holds for $k[X_1, \dots, X_n]$. Let $F \in k[X_1, \dots, X_n, X_{n+1}] \simeq k[X_1, \dots, X_n][X_{n+1}]$ satisfying $F(a_1, ..., a_n) = 0$ for all $a_1, ..., a_n \in k$. We call $f_i \in k[X_1, \dots, X_n]$ the coefficient of X_{n+1}^i, $0 \leq i \leq \deg_{X_{n+1}} F$. For all $(a_1, \dots, a_n) \in k^n$, $F(a_1, \dots, a_n, X_{n+1})$, is a polynomial in one variable which vanishes at all $a_{n+1} \in k$, so it is 0. Thus for all $(a_1, \dots, a_n) \in k^n$, $f_i(a_1, \dots, a_n) = 0$. By induction hypothesis, $f_i = 0$ so $F = 0$.

2. $X^p - X$ in $\mathbb{F}_p[X]$ is zero at all element of $\mathbb{F}_p$. But it is not zero in the polynomial ring.

Exercise 5. Let $F \in k[X_1, \dots, X_n]$ be homogeneous of degree d. Show that:

$$\sum_{i=1}^n X_i \cdot \frac{\partial F}{\partial X_i} = d \cdot F$$

Solution 5.

By linearity, It suffices to prove it for a monomial $F = X_1^{i_1} \dots X_n^{i_n}$. Then, given $j \in \{1, \dots, n\}$,

$$\frac{\partial F}{\partial X_j} = i_j X_1^{i_1} \dots X_j^{i_j - 1} \dots X_n^{i_n}$$

$$X_j \cdot \frac{\partial F}{\partial X_j} = i_j F$$

$$\sum_{j=1}^{n} X_i \cdot \frac{\partial F}{\partial X_j} = (\sum_{i=1}^{n} i_j) \cdot F$$

If F has degree d then $\sum_{j=1}^{n} i_j = d$ and we are done.